

SOME RADICALS, FRATTINI AND CARTAN SUBALGEBRAS OF LEIBNIZ n -ALGEBRAS

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ABSTRACT. In the present work we introduce notions such as k -solvability, s - and K_1 -nilpotency and the corresponding radicals. We prove that these radicals are invariant under derivations of Leibniz n -algebras. The Frattini and Cartan subalgebras of Leibniz n -algebras are studied. In particular, we construct examples that show that a classical result on conjugacy of Cartan subalgebras of Lie algebras, which also holds in Leibniz algebras and Lie n -algebras, is not true for Leibniz n -algebras.

1. INTRODUCTION

This work is devoted to the investigation of Leibniz n -algebras. In 1985, Filippov [11] introduced a notion of Lie n -algebra with an n -ary skew-symmetric multiplication which satisfies the identity

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n] \quad (1)$$

bearing in mind the general notion of Ω -algebra considered by Kurosh [14].

Earlier in 1973, Nambu [18] had constructed an example of 3-Lie algebra, where the multiplication for a triple of classical observables on the three-dimensional phase space \mathbb{R}^3 was given by the Jacobian. This bracket naturally generalizes the usual Poisson bracket from a binary to a ternary operation.

In 1993, Loday [15, 16] introduced a non skew-symmetric version of Lie algebras, the so-called Leibniz algebras. As a generalization of Leibniz algebras and n -Lie algebras, in 2002, Casas, Loday and Pirashvili [10] defined n -Leibniz algebras as a non skew-symmetric version of Lie n -algebras. They also presented constructions between the varieties of Leibniz algebras and Leibniz n -algebras ($n \geq 3$) which are not invertible.

In the present work, in Section 2, we introduce the Frattini subalgebra of a Leibniz n -algebra and establish properties extending some results of Leibniz algebras and of Lie n -algebras. Frattini theory was originally discovered in group theory and further have been studied in Lie algebras in [17, 6, 20], in Lie n -algebras in [4, 22] and in Leibniz algebras in [7, 8]. Here we show that many results concerning Frattini subalgebras and Frattini ideals from the theory of Lie n -algebras remain true when we omit the skew-symmetrical property of the n -ary multiplication.

In Section 3, we study the right multiplication operators in a Leibniz n -algebra. Filippov [11] noted that the so-called right multiplication operators play the same crucial role in the theory of Lie n -algebras as in Lie algebras since they form a Lie algebra with respect to the commutator. The space of the right multiplication operators in Leibniz n -algebras also forms an ideal in the Lie algebra of derivations. However, in the case $n \geq 3$, some well-known properties of the right multiplication operators do not hold in

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general; for instance, in [2] it was given an example of a Leibniz n -algebra which admits a non-degenerate right multiplication operator. Because of that curious properties of these operators, to obtain some results on right multiplication operators which are valid for Leibniz and Lie n -algebras we must consider them with additional conditions.

In Section 4, we study solvability and nilpotency in Leibniz n -algebras and show that the solvable and nilpotent radicals are invariant under all derivations. Since multiplication in Leibniz n -algebras is not anti-symmetric in all the variables, notions such as nilpotency and solvability may be introduced in different ways depending on the position of the multiplicand. The product in the definition of the corresponding series is not necessarily an ideal and this makes some arguments difficult to prove. Hence, we introduce special notions, as k -solvability, nilpotency and K_1 -nilpotency of Leibniz n -algebras. Most of them agree with the corresponding notions on particular cases: Lie n -algebras [13] and Leibniz algebras. We establish some properties of k -solvable (nilpotent) ideals, as well.

Finally, in Section 5, we construct examples that show the non-conjugacy of Cartan subalgebras for Leibniz n -algebras. In [2], it was proved that the null root subspace of the right multiplication operators with respect to a regular element is a nilpotent subalgebra. Here we obtain that this subalgebra under some restriction is a Cartan subalgebra. Moreover, a classical result about conjugacy of Cartan subalgebras in Lie algebras that was extended to the general cases - Leibniz algebras [19] and Lie n -algebras [13], unfortunately does not hold in the case of Leibniz n -algebras ($n \geq 2$). Starting with a particular Lie n -algebra, we construct Leibniz n -algebras which factored out by the ideal I generated by the elements $[x_1, \dots, x_i, \dots, x_j, \dots, x_n]$, where $x_i = x_j$ for some $1 \leq i \neq j \leq n$, are isomorphic to the given Lie n -algebra under some conditions. These Leibniz n -algebras have Cartan subalgebras of different dimensions and therefore they are not conjugated (see Example 5.7).

2. PRELIMINARIES

Definition 2.1 ([10]). *A vector space L with an n -ary multiplication $[-, -, \dots, -] : L^{\otimes n} \rightarrow L$ is called a Leibniz n -algebra if it satisfies the following identity*

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n] \quad (1)$$

It should be noted that if the product $[-, -, \dots, -]$ is skew-symmetric in each pair of variables, i.e.

$$[x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_n] = -[x_1, x_2, \dots, x_j, \dots, x_i, \dots, x_n],$$

then this Leibniz n -algebra becomes a Lie n -algebra.

Since in Leibniz n -algebras the n -ary multiplication is not necessarily skew-symmetrical, basic notions such as ideals have to be considered with additional conditions.

Definition 2.2. *A subspace I of a Leibniz n -algebra L is called an s -sided ideal of L , if*

$$[\underbrace{L, \dots, L}_{s-1}, I, \underbrace{L, \dots, L}_{n-s}] \subseteq I.$$

If I is s -ideal for all $1 \leq s \leq n$, then I is called an ideal.

Definition 2.3. *A proper subalgebra M of a Leibniz n -algebra L is called maximal if the only subalgebra properly containing M is L .*

Definition 2.4. The intersection of all maximal subalgebras of a Leibniz n -algebra L is a subalgebra denoted by $F(L)$ and it is called the Frattini subalgebra.

The maximal ideal of L that is contained in $F(L)$ is called the Frattini ideal and it is denoted by $\phi(L)$.

The following statements which hold for Lie n -algebras [4] can be extended in a similar way to the case of Leibniz n -algebras.

Proposition 2.5. Let L be a Leibniz n -algebra. Then the following statements hold:

- (1) If B is a subalgebra of L such that $B + F(L) = L$, then $B = L$.
- (2) If B is a subalgebra of L such that $B + \phi(L) = L$, then $B = L$.

Proposition 2.6. Let L be a Leibniz n -algebra and B an ideal of L . Then there exists a proper subalgebra C of L such that $L = B + C$ iff $B \not\subseteq F(L)$.

Moreover, the assertion of Proposition 2.6 holds if we substitute $\phi(L)$ for $F(L)$.

Proposition 2.7. Let C be a subalgebra of L and B an ideal of L such that $B \subseteq F(C)$ ($B \subseteq \phi(C)$).

Then $B \subseteq F(L)$ ($B \subseteq \phi(L)$, respectively).

Corollary 2.8. Let L be a Leibniz n -algebra and B a subalgebra of L such that $F(B)$ ($\phi(B)$) is an ideal of L . Then $F(B) \subseteq F(L)$ ($\phi(B) \subseteq \phi(L)$, respectively).

Proposition 2.9. Let L be a Leibniz n -algebra and B an ideal of L . Then the following statements hold:

- (1) $(F(L) + B)/B \subseteq F(L/B)$, $((\phi(L) + B)/B \subseteq \phi(L/B))$;
- (2) If $B \subseteq F(L)$ then $F(L)/B = F(L/B)$, $\phi(L)/B = \phi(L/B)$;
- (3) If $F(L/B) = 0$ ($\phi(L/B) = 0$), then $F(L) \subseteq B$ ($\phi(L) \subseteq B$).

Theorem 2.10. If a Leibniz n -algebra L has a decomposition

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_m,$$

where L_i ($1 \leq i \leq m$) are ideals of L , then

- (1) $F(L) \subseteq F(L_1) + \cdots + F(L_m)$;
- (2) $\phi(L) = \phi(L_1) + \cdots + \phi(L_m)$.

Given an arbitrary Leibniz n -algebra L consider the following sequences (s is a fixed natural number, $1 \leq s \leq n$):

$$L^{<1>s} = L, \quad L^{<k+1>s} = [\underbrace{L, \dots, L}_{(s-1)\text{-times}}, L^{<k>s}, \underbrace{L, \dots, L}_{(n-s)\text{-times}}],$$

$$L^1 = L, \quad L^{k+1} = \sum_{i=1}^n [\underbrace{L, \dots, L}_{(i-1)\text{-times}}, L^k, \underbrace{L, \dots, L}_{(n-i)\text{-times}}].$$

Definition 2.11. A Leibniz n -algebra L is said to be s -nilpotent (nilpotent) if there exists a natural number $k \in \mathbb{N}$ ($l \in \mathbb{N}$) such that $L^{<k>s} = 0$ ($L^l = 0$, respectively).

It should be noticed that for Lie n -algebras the above notions of s -nilpotency and nilpotency coincide. Recall also that for Leibniz algebras (i.e. Leibniz 2-algebras) the notions of 1-nilpotency and nilpotency also coincide [3].

In [2, Example 2.2], it is shown that the s -nilpotency property for Leibniz n -algebra ($n \geq 3$) essentially depends on s .

Let H be an ideal of a Leibniz n -algebra L . Put $H^{(1)k} = H$ and

$$H^{(m+1)k} = \sum_{i_1 + \dots + i_k = 0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, H^{(m)k}, \underbrace{L, \dots, L}_{i_2}, H^{(m)k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)k}, \underbrace{L, \dots, L}_{n-i_1-\dots-i_k}]$$

for all $1 \leq k \leq n$ and $m \geq 1$.

Definition 2.12. An n -sided ideal H of Leibniz n -algebra is said to be k -solvable with index of k -solvability equal to m if there exists $m \in \mathbb{N}$ such that $H^{(m)k} = 0$ and $H^{(m-1)k} \neq 0$.

When $L = H$, L is called a k -solvable Leibniz n -algebra.

Notice that this definition agrees with the definition of k -solvability of Lie n -algebras given in [13].

Definition 2.13. We say that a subalgebra U of a Leibniz n -algebra L is left subnormal if there exists a chain of subalgebras $U = U_k \subseteq \dots \subseteq U_1 \subseteq U_0 = L$ with each U_{i+1} an r -ideal ($r \neq 1$) in U_i .

Theorem 2.14. Let U be a left subnormal subalgebra of Leibniz n -algebra L and V an ideal in U such that $V \subseteq F(L)$. If U/V is 1-nilpotent, then U is 1-nilpotent.

Proof. Similar to the proof of [7, Theorem 3.6]. □

The following statements hold for Lie n -algebras [4] and are also true for Leibniz n -algebras.

Corollary 2.15. If $I \subseteq F(L)$ is an r -ideal ($r \neq 1$) of L , then I is 1-nilpotent. Particularly, $\phi(L)$ is a 1-nilpotent ideal of L .

Definition 2.16. In a Leibniz n -algebra L the intersection of all maximal ideals of L is called the Jacobson radical and it is denoted by $J(L)$.

Proposition 2.17. Let L be a finite dimensional Leibniz n -algebra. Then

$$F(L) \subseteq [L, L, \dots, L] \text{ and } J(L) \subseteq [L, L, \dots, L].$$

Moreover, if L is a k -solvable Leibniz n -algebra, then

$$J(L) = [L, L, \dots, L].$$

Theorem 2.18. Let L be a finite dimensional nilpotent Leibniz n -algebra. Then the following statements hold:

- (1) Any maximal subalgebra M of L is an ideal of L ;
- (2) $F(L) = \phi(L) = J(L) = [L, L, \dots, L]$.

3. RIGHT MULTIPLICATION OPERATORS

Definition 3.1. A linear map d defined on a Leibniz n -algebra L is called a derivation if

$$d([x_1, x_2, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n].$$

The space of all derivations of a given Leibniz n -algebra L is denoted by $\text{Der}(L)$.

The space $\text{Der}(L)$ forms a Lie algebra with respect to the commutator [2].

Set $A^{\times k} = \underbrace{A \times A \times \cdots \times A}_{k\text{-times}}$.

Given an arbitrary element $x = (x_2, \dots, x_n) \in L^{\times(n-1)}$ consider the operator $R(x) : L \rightarrow L$ of right multiplication defined by

$$R(x)(z) = [z, x_2, \dots, x_n].$$

Any right multiplication operator is a derivation and the space $R(L)$ of all right multiplication operators forms a Lie ideal of $\text{Der}(L)$ [2].

Theorem 3.2 ([2] Engel's theorem). *A Leibniz n -algebra L is 1-nilpotent if and only if $R(x)$ is nilpotent for all $x \in L^{\times(n-1)}$.*

In [2] it was given an example of a Leibniz n -algebra which admits a non-degenerated right multiplication operator. This is the significant difference between Leibniz n -algebras ($n \geq 3$) on one hand and Leibniz algebras and Lie n -algebras on the other.

Below we assume that all right multiplication operators are degenerated.

The following lemma yields a decomposition of a given vector space into a direct sum of two subspaces which are invariant with respect to a given linear transformation.

Lemma 3.3 (Fitting Lemma). *Let V be a vector space and $A : V \rightarrow V$ be a linear transformation. Then $V = V_{0A} \oplus V_{1A}$, where $A(V_{0A}) \subseteq V_{0A}$, $A(V_{1A}) \subseteq V_{1A}$ and $V_{0A} = \{v \in V \mid A^i(v) = 0 \text{ for some } i\}$ and $V_{1A} = \bigcap_{i=1}^{\infty} A^i(V)$. Moreover, $A|_{V_{0A}}$ is a nilpotent transformation and $A|_{V_{1A}}$ is an automorphism. V_{0A} is called the Fitting null-component of V with respect to A .*

Proof. See [12, Chapter II, §4]. □

Definition 3.4. *An element $h \in L^{\times(n-1)}$ is said to be regular for the algebra L if the dimension of the Fitting null-component of the space L with respect to $R(h)$ is minimal.*

Lemma 3.5 ([9]). *Let L be a finite dimensional complex Leibniz n -algebra with given derivation d , and let $L = L_{\alpha} \oplus L_{\beta} \oplus \cdots \oplus L_{\gamma}$ be the decomposition of the algebra L into root spaces with respect to the derivation d (i.e. $L_{\alpha} = \{x \in L \mid (d - \alpha I)^k x = 0 \text{ for some } k\}$). Then*

$$[L_{\alpha_1}, L_{\alpha_2}, \dots, L_{\alpha_n}] \subseteq \begin{cases} 0 & \text{if } \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ is not a root of } d \\ L_{\alpha_1 + \alpha_2 + \cdots + \alpha_n} & \text{if } \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ is a root of } d. \end{cases}$$

Proposition 3.6. *In a Leibniz n -algebra L any right multiplication operator $R(a_2, \dots, a_n)$ is a sum of right multiplication operators with zero root space with respect to $R(a_2, \dots, a_n)$.*

Proof. Let $\alpha_0 = 0, \alpha_1, \dots, \alpha_k$ be the eigenvalues of $R(a_2, \dots, a_n)$. Then L is decomposed into a direct sum

$$L = L_0 \oplus L_{\alpha_1} \oplus \cdots \oplus L_{\alpha_k},$$

where $L_{\alpha_i} = \{x \mid (R(a_2, \dots, a_n) - \alpha_i I)^m(x) = 0 \text{ for some } m \in \mathbb{N}\}$.

Consider $a_i = a_0^i + a_{\alpha_1}^i + \cdots + a_{\alpha_k}^i$, $a_{\alpha_m}^i \in L_m$, $2 \leq i \leq k$. Then for all $x \in L$, we have

$$\begin{aligned} R(a_2, \dots, a_n)(x) &= [x, a_2, \dots, a_n] = [x, a_0^2 + a_{\alpha_1}^2 + \cdots + a_{\alpha_k}^2, \dots, a_0^n + a_{\alpha_1}^n + \cdots + a_{\alpha_k}^n] \\ &= [x, a_0^2, \dots, a_0^n] + [x, a_{\alpha_1}^2, a_0^3, \dots, a_0^n] + \cdots + [x, a_{\alpha_k}^2, a_{\alpha_k}^3, \dots, a_{\alpha_k}^n] \\ &= R(a_0^2, a_0^3, \dots, a_0^n)(x) + R(a_{\alpha_1}^2, a_0^3, \dots, a_0^n)(x) + \cdots + R(a_{\alpha_k}^2, a_{\alpha_k}^3, \dots, a_{\alpha_k}^n)(x). \end{aligned}$$

By Lemma 3.5, we obtain that $R(a_2, \dots, a_n)(x) = B(x) + C(x)$, where B is a sum of right multiplication operators with zero weight and C is a sum of right multiplication operators with nonzero weights. Then for any $x \in L_{\alpha_i}$, we have

$$C(x) = (R(a_2, \dots, a_n) - B)(x) = R(a_2, \dots, a_n)(x) - B(x) \subseteq L_{\alpha_i},$$

which holds only if $C(x) = 0$ since C adds a weight. Therefore, C is a zero operator on L_{α_i} . Since $L = L_0 \oplus L_{\alpha_1} \oplus \dots \oplus L_{\alpha_k}$, we obtain $C = 0$ on L .

So, $R(a_2, \dots, a_n) = B$, i.e. is a sum of right multiplication operators with zero weight with respect to $R(a_2, \dots, a_n)$. \square

In [7] it was proved the following result for left Leibniz algebras which is also valid for right Leibniz algebras, i.e. Leibniz 2-algebras.

Lemma 3.7 ([7]). *In a Leibniz algebra L for any $a \in L$ there exists $b \in L_0(R_a)$ such that $L_0(R_b) = L_0(R_a)$.*

Concerning this lemma we establish the following result for the case $n \geq 3$.

Corollary 3.8. *If the nonzero eigenvalues $\alpha_1, \dots, \alpha_k$ of the right multiplication operator $R(a_2, \dots, a_n)$ in a Leibniz n -algebra ($n \geq 3$) satisfy*

$$\mu_1\alpha_1 + \mu_2\alpha_2 + \dots + \mu_k\alpha_k \neq 0,$$

for all non-negative integers μ_1, \dots, μ_k such that

$$0 < \mu_1 + \dots + \mu_k \leq n - 1,$$

then there exist $b_2, b_3, \dots, b_n \in L_0(R(a_2, \dots, a_n))$ such that

$$L_0(R(b_2, \dots, b_n)) = L_0(R(a_2, \dots, a_n)).$$

Proof. From Proposition 3.6 we obtain that $R(a_2, \dots, a_n) = B$. From the condition on the eigenvalues we conclude that B consists of just one right multiplication operator, namely $B = R(a_0^2, a_0^3, \dots, a_0^n)$. So, if we take $b_i = a_0^i$, we obtain $L_0(R(b_2, \dots, b_n)) = L_0(R(a_2, \dots, a_n))$. \square

A Leibniz n -algebra satisfying the conditions of Corollary 3.8 is given in the following

Example 3.9 ([2]). *Consider a Leibniz n -algebra $L = \langle e_1, e_2, \dots, e_n \rangle$ with the following multiplication:*

$$[e_k, e_1, \dots, e_1] = e_k \quad (2 \leq k \leq m).$$

The right multiplication operator $R(e_1, \dots, e_1)$ has only two eigenvalues: 0 and 1. It is easy to see that the conditions of Corollary 3.8 are satisfied and $e_1 \in L_0(R(e_1, \dots, e_1))$.

Below, we present an example which shows the sufficiency of the condition in Corollary 3.8.

Example 3.10. *Consider an m dimensional Leibniz n -algebra L with the following multiplication:*

$$\begin{aligned} [e_k, e_1, e_2, \dots, e_{n-1}] &= \alpha_k e_k \\ [e_{k+1}, e_1, e_2, \dots, e_{n-1}] &= \alpha_{k+1} e_{k+1} \\ &\vdots \\ [e_m, e_1, e_2, \dots, e_{n-1}] &= \alpha_m e_m \end{aligned}$$

where $\{e_1, \dots, e_m\}$ is a basis, $k < n - 1$ and $\sum_{i=k}^{n-1} \alpha_i = 0, \alpha_k \cdots \alpha_m \neq 0$.

Then $L_0(R(e_1, \dots, e_{n-1})) = \{e_1, \dots, e_{n-1}\}$. Since any other right multiplication operator either coincides with $R(e_1, \dots, e_{n-1})$ or is identically zero, there does not exist $b_2, \dots, b_n \in L_0(R(e_1, \dots, e_{n-1}))$ such that $L_0(R(b_2, \dots, b_n)) = L_0(R(e_1, \dots, e_{n-1}))$.

Definition 3.11 ([2]). Given a subset X in a Leibniz n -algebra, the s -normalizer of X is the set

$$N_s(X) = \{a \in L \mid [x_1, \dots, x_{s-1}, a, x_{s+1}, \dots, x_n] \in X \text{ for all } x_i \in X\}.$$

The set $N(X) = \bigcap_{s=1}^n N_s(X)$ is called the normalizer of X .

Notice that, if X is a subalgebra of L , then $N(X), N_s(X) \supseteq X$.

Lemma 3.12 ([2]). Let M be an invariant subspace of a vector space L with respect to a linear transformation $Q : L \rightarrow L$. Let $x = x_0 + x_\alpha + x_\beta + \dots + x_\gamma$ be any decomposition of an element x into a sum of characteristic vectors from the corresponding characteristic spaces $L_\xi (\xi \in \{0, \alpha, \beta, \dots, \gamma\})$. If $Q(x) \in M$, then $x - x_0 \in M$.

The following lemma is an extension of [7, Lemma 3.2] under the condition $a_2, \dots, a_n \in L_0(R(a_2, \dots, a_n))$.

Lemma 3.13. Let L be a Leibniz n -algebra and $R(a_2, \dots, a_n) : L \rightarrow L$ a right multiplication operator such that $a_2, \dots, a_n \in L_0(R(a_2, \dots, a_n))$. Then for any subalgebra U containing $L_0(R(a_2, \dots, a_n))$ the equality $N(U) = U$ holds.

Proof. Let $z \in N_1(U)$. Then $[z, U, \dots, U] \subseteq U$. Denote $L_0 = L_0(R(a_2, \dots, a_n))$. Then

$$R(a_2, \dots, a_n)(z) = [z, a_2, \dots, a_n] \in [z, L_0, \dots, L_0] \subseteq [z, U, \dots, U] \subseteq U.$$

Hence $R(a_2, \dots, a_n)(z) \in U$.

Notice that $R(a_2, \dots, a_n)(U) = [U, a_2, \dots, a_n] \subseteq [U, L_0, \dots, L_0] \subseteq [U, U, \dots, U] \subseteq U$ since U is a subalgebra. Therefore, the conditions of Lemma 3.12 are satisfied. Thus $z - z_0 \in U$. Then $z \in U$. So we have proved $N_1(U) = U$.

Since U is a subalgebra, $N_s(U) \supseteq U$ for all $2 \leq s \leq n$. Then $N(U) = N_1 \cap (\bigcap_{s=2}^n N_s(U)) = U$. \square

Proposition 3.14. Let a_2, \dots, a_n be elements of a Leibniz n -algebra L such that $a_2, \dots, a_n \in L_0(R(a_2, \dots, a_n))$. If every maximal subalgebra is an i - and a j -ideal ($1 \leq i \neq j \leq n$) in L , then $R(a_2, \dots, a_n)$ is nilpotent.

Proof. Assume that $L_0(R(a_2, \dots, a_n)) \neq L$. Then there exists maximal algebra M such that $L_0(R(a_2, \dots, a_n)) \subseteq M$. Then by previous lemma we have $N(M) = M$.

Since M is an i - and a j -ideal ($i \neq j$), we have

$$\underbrace{[M, \dots, M]_{s-1}, L, M, \dots, M}_{s-1} \subseteq M$$

for all $1 \leq s \leq n$. Thus, $L = N_s(M)$ for all $1 \leq s \leq n$ and $L = N(M)$. Contradiction.

Therefore $L = L_0(R(a_2, \dots, a_n))$ and $R(a_2, \dots, a_n)$ is a nilpotent operator. \square

In [21, Theorem 2.2] there were given several statements equivalent to nilpotency of the finite dimensional Lie n -algebras. For Leibniz n -algebras Proposition 2.17 verifies the statement in one direction. The other direction of the statement in our case is not true in general. However we establish the following result.

Proposition 3.15. *Let L be a finite dimensional Leibniz n -algebra with condition $a_i \in L_0(R(a_2, \dots, a_n))$ for some $2 \leq i \leq n$ for an arbitrary $(a_2, \dots, a_n) \in L^{\times(n-1)}$. If any maximal subalgebra M of L is an ideal of L then L is 1-nilpotent.*

Proof. Assume that L is not 1-nilpotent. Then there exists a non-nilpotent right multiplication operator $R(a_2, \dots, a_n)$. Since $R(a_2, \dots, a_n)$ is non-nilpotent, the Fitting null-component $L_0(R(a_2, \dots, a_n)) \neq L$.

Let M be a maximal subalgebra of L containing $L_0(R(a_2, \dots, a_n))$. Then $a_i \in L_0(R(a_2, \dots, a_n)) \subseteq M$ for some $2 \leq i \leq n$ by assumption of the proposition. Since M is a maximal subalgebra, it is also an ideal of L . Then $R(a_2, \dots, a_n)(L) \subseteq M$.

Since $R(a_2, \dots, a_n)$ is an automorphism on $L_1(R(a_2, \dots, a_n))$, we obtain that $L_1 = R(L_1) = L_1 \cap M$. Hence $L_1 \subseteq M$.

Then $L = L_0 \oplus L_1 \subseteq M \neq L$. This is a contradiction. Hence, all right multiplication operators are nilpotent. Therefore, by Engel's theorem L is 1-nilpotent. \square

4. INVARIANCE OF SOME RADICALS UNDER DERIVATION

In the following section we establish some classical results from the theory of Lie algebras concerning solvability and nilpotency which are also true in Leibniz algebras and Lie n -algebras.

Proposition 4.1. *For an ideal H of a Leibniz n -algebra L the equality $(H^{(m)_k})^{(r)_k} = H^{(m+r-1)_k}$ holds for all $m, r \in \mathbb{N}$.*

Proof. Using induction on r one can easily prove the assertion of the proposition. \square

Even though we can not state that H^{m_k} is an s -sided ideal for all $1 \leq s \leq n$, we establish the following result.

Proposition 4.2. *For an ideal H of a Leibniz n -algebra L , $H^{(m)_k}$ is a 1-ideal of L for all $m, k \in \mathbb{N}$.*

Proof. Let k be an arbitrary fixed natural number.

For $m = 1$ we have $[H^{(1)_k}, L, \dots, L] \subseteq [H, L, \dots, L] \subseteq H = H^{(1)_k}$ since H is an ideal. Let $H^{(m)_k}$ be a 1-ideal, i.e. $[H^{(m)_k}, L, \dots, L] \subseteq H^{(m)_k}$.

Then

$$\begin{aligned} & [H^{(m+1)_k}, L, \dots, L] \\ &= \left[\sum_{i_1 + \dots + i_k = 0}^{n-k} \underbrace{[L, \dots, L]}_{i_1}, H^{(m)_k}, \dots, \underbrace{[L, \dots, L]}_{i_k}, H^{(m)_k}, \underbrace{[L, \dots, L]}_{n-i_1-\dots-i_k}, L, \dots, L \right] \\ &= \sum_{i_1 + \dots + i_k = 0}^{n-k} \left[\underbrace{[L, \dots, L]}_{i_1}, H^{(m)_k}, \dots, \underbrace{[L, \dots, L]}_{i_k}, H^{(m)_k}, \underbrace{[L, \dots, L]}_{n-i_1-\dots-i_k}, L, \dots, L \right]. \end{aligned}$$

Since $H^{(m)_k}$ is a 1-ideal by induction hypothesis, using identity (1) we obtain that

$$\begin{aligned} & \left[\underbrace{[L, \dots, L]}_{i_1}, H^{(m)_k}, \dots, \underbrace{[L, \dots, L]}_{i_k}, H^{(m)_k}, \underbrace{[L, \dots, L]}_{n-i_1-\dots-i_k}, L, \dots, L \right] \\ & \subseteq \underbrace{[L, \dots, L]}_{i_1}, H^{(m)_k}, \dots, \underbrace{[L, \dots, L]}_{i_k}, H^{(m)_k}, \underbrace{[L, \dots, L]}_{n-i_1-\dots-i_k}. \end{aligned}$$

Therefore

$$\begin{aligned} & [H^{(m+1)_k}, L, \dots, L] \\ & \subseteq \sum_{i_1 + \dots + i_k = 0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, H^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, H^{(m)_k}, \underbrace{L, \dots, L}_{n-i_1-\dots-i_k}] = H^{(m+1)_k} \end{aligned}$$

and $H^{(m+1)_k}$ is a 1-ideal of L . \square

Proposition 4.3. *Let I be a k -solvable ideal of a Leibniz n -algebra L such that L/I is also k -solvable. Then L is k -solvable.*

Proof. Let $\phi : L \rightarrow L/I$ be the natural homomorphism. Since L/I is k -solvable, we have $0 = (L/I)^{(m)_k} = (\phi(L))^{(m)_k} = \phi(L^{(m)_k})$ for some $m \in \mathbb{N}$. Thus $L^{(m)_k} \subseteq I$. Since I is k -solvable, there exists $p \in \mathbb{N}$ such that $I^{(p)_k} = 0$. Therefore by Proposition 4.1 we have $L^{(m+p-1)_k} = (L^{(m)_k})^{(p)_k} \subseteq I^{(p)_k} = 0$ and so L is k -solvable. \square

By induction it is easy to prove that if I is a k -solvable ideal of a Leibniz n -algebra L , then I is also $(k+p)$ -solvable for all $p \in \mathbb{N}$.

Using standard methods and Proposition 4.3 we obtain that the sum of k -solvable ideals is also k -solvable. Now let H be a maximal k -solvable ideal in a finite dimensional Leibniz n -algebra L and let K be an arbitrary k -solvable ideal of L . Then $H + K$ is also k -solvable and $H + K \supseteq H$. Since H is a maximal k -solvable ideal, we obtain that $H + K = H$. Therefore we can define the maximal k -solvable ideal as the sum of all the k -solvable ideals in L and call it the *k -solvable radical*.

The following formula for a derivation $d : L \rightarrow L$ of a Leibniz n -algebra L over a field \mathbb{K} of characteristic zero, for any $k \in \mathbb{N}$, was given in [9]:

$$d^k([x_1, \dots, x_n]) = \sum_{i_1 + i_2 + \dots + i_n = k} \frac{k!}{i_1! i_2! \dots i_n!} [d^{i_1}(x_1), d^{i_2}(x_2), \dots, d^{i_n}(x_n)]. \quad (2)$$

Proposition 4.4. *Let I be an ideal of a Leibniz n -algebra L and $d \in \text{Der}(L)$. Then*

$$(d(I))^{(m)_k} \subseteq I + d^{k^{m-1}}(I^{(m)_k})$$

for all $m \in \mathbb{N}$ and $1 \leq k \leq n$.

Proof. For $m = 1$ we have $d(I) \subseteq I + d(I)$ which obviously holds.

Assume that $(d(I))^{(m)_k} \subseteq I + d^{k^{m-1}}(I^{(m)_k})$.

Using formula (2) we verify the inclusion for $m + 1$:

$$\begin{aligned} & (d(I))^{(m+1)_k} \\ & = \sum_{i_1 + \dots + i_k = 0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, d(I)^{(m)_k}, \underbrace{L, \dots, L}_{i_2}, d(I)^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, d(I)^{(m)_k}, \underbrace{L, \dots, L}_{n-i_1-\dots-i_k}] \\ & \subseteq \sum_{i_1 + \dots + i_k = 0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, I + d^{k^{m-1}}(I^{(m)_k}), \dots, \underbrace{L, \dots, L}_{i_k}, I + d^{k^{m-1}}(I^{(m)_k}), \underbrace{L, \dots, L}_{n-i_1-\dots-i_k}] \\ & \subseteq I + d^{k^m} \left(\sum_{i_1 + \dots + i_k = 0}^{n-k} [\underbrace{L, \dots, L}_{i_1}, I^{(m)_k}, \dots, \underbrace{L, \dots, L}_{i_k}, I^{(m)_k}, \underbrace{L, \dots, L}_{n-i_1-\dots-i_k}] \right) = I + d^{k^m}(I^{(m+1)_k}). \end{aligned}$$

Therefore, the assertion of the proposition is true. \square

Also, in [9], it was shown that for any ideal I of L and $d \in \text{Der}(L)$ the $I + d(I)$ is also an ideal of L .

Theorem 4.5. *Let J be the k -solvable radical of a finite dimensional Leibniz n -algebra L over a field \mathbb{K} of characteristic zero. Then $d(J) \subseteq J$ for any $d \in \text{Der}(L)$.*

Proof. Let $s \in \mathbb{N}$ be such $J^{(s)k} = 0$. Then by Proposition 4.4 we have $(d(J))^{(s)k} \subseteq J + d^{k^{s-1}}(J^{(s)k}) = J$.

Using formula (2), we obtain that $(J + d(J))^{(s)k} \subseteq J + (d(J))^{(s)k} \subseteq J$. Now by Proposition 4.1 we have $(J + d(J))^{(2s-1)k} = \left((J + d(J))^{(s)k} \right)^{(s)k} \subseteq J^{(s)k} = 0$. But this means that $J + d(J)$ is a k -solvable ideal. Since J is a k -solvable radical, we obtain that $J + d(J) \subseteq J$ and therefore $d(J) \subseteq J$. \square

Similarly as in [5] we introduce the following series for a 1-sided ideal I of a Leibniz n -algebra L :

$$I^{[1]} = I, \quad I^{[k+1]} = [I^{[k]}, I, L, \dots, L] \quad (k \geq 1).$$

By a simple induction using identity (1) it can be proved that for any 1-sided ideal I and for all $n \in \mathbb{N}$, $I^{[n]}$ is a 1-sided ideal.

Definition 4.6. *A 1-sided ideal I is called K_1 -nilpotent, if there exists $k \in \mathbb{N}$ such that $I^{[k]} = 0$.*

The introduced type of nilpotency is also known as nilpotency in the sense of Kuzmin for Lie n -algebras. Identity (1) is organized in such way, that the elements of the above introduced series are 1-ideals. However, if we change the position of $I^{[k]}$ in the product defining $I^{[k+1]}$ from the first to any other, we are not able to state that the elements of the obtained series will be s -ideals of L for any $2 \leq s \leq n$.

Proposition 4.7. *Let I and J be K_1 -nilpotent 1-sided ideals. Then $I + J$ is also a K_1 -nilpotent 1-sided ideal.*

Proof. First, observe that

$$[I^{[p]} \cap J^{[q]}, I, L, \dots, L] \subseteq [I^{[p]}, I, L, \dots, L] = I^{[p+1]},$$

and since $J^{[q]}$ is a 1-ideal, we get

$$[I^{[p]} \cap J^{[q]}, I, L, \dots, L] \subseteq [J^{[q]}, I, L, \dots, L] \subseteq J^{[q]}.$$

Therefore,

$$[I^{[p]} \cap J^{[q]}, I, L, \dots, L] \subseteq I^{[p+1]} \cap J^{[q]}.$$

Analogously,

$$[I^{[p]} \cap J^{[q]}, J, L, \dots, L] \subseteq I^{[p]} \cap J^{[q+1]}.$$

We have $(I + J)^{[1]} = I + J = I^{[1]} + J^{[1]}$.

Now assume that

$$(I + J)^{[k]} \subseteq I^{[k]} + (I^{[k-1]} \cap J^{[1]}) + \dots + (I^{[1]} \cap J^{[k-1]}) + J^{[k]}.$$

Then

$$\begin{aligned}
(I + J)^{[k+1]} &= [(I + J)^{[k]}, I + J, L, \dots, L] \\
&\subseteq [(I + J)^{[k]}, I, L, \dots, L] + [(I + J)^{[k]}, J, L, \dots, L] \\
&\subseteq [I^{[k]}, I, L, \dots, L] + \sum_{r=1}^{k-1} [I^{[k-r]} \cap J^{[r]}, I, L, \dots, L] + [J^{[k]}, I, L, \dots, L] \\
&\quad + [I^{[k]}, J, L, \dots, L] + \sum_{r=1}^{k-1} [I^{[k-r]} \cap J^{[r]}, J, L, \dots, L] + [J^{[k]}, J, L, \dots, L] \\
&\subseteq I^{[k+1]} + \left(\sum_{r=1}^{k-1} I^{[k-r+1]} \cap J^{[r]} \right) + (I^{[1]} \cap J^{[k]}) \\
&\quad + (I^{[k]} \cap J^{[1]}) + \left(\sum_{r=1}^{k-1} I^{[k-r]} \cap J^{[r+1]} \right) + J^{[k+1]} \\
&\subseteq I^{[k+1]} + (I^{[k]} \cap J^{[1]}) + \dots + (I^{[1]} \cap J^{[k]}) + J^{[k+1]}.
\end{aligned}$$

Hence, for any $n \in \mathbb{N}$ we have

$$(I + J)^{[n]} \subseteq I^{[n]} + (I^{[n-1]} \cap J^{[1]}) + \dots + (I^{[1]} \cap J^{[n-1]}) + J^{[n]}.$$

So if $I^{[n_1]} = 0$ and $J^{[n_2]} = 0$, then for $n = n_1 + n_2$ every summand in the above sum is zero. Therefore $(I + J)$ is also K_1 -nilpotent. \square

Corollary 4.8. *Let I and J be K_1 -nilpotent ideals. Then $I + J$ is also a K_1 -nilpotent ideal.*

Let I be a maximal K_1 -nilpotent ideal in a finite dimensional Leibniz n -algebra L and let J be an arbitrary K_1 -nilpotent ideal of L . Then $I + J$ is also K_1 -nilpotent and $I + J \supseteq I$. Since I is a maximal K_1 -nilpotent ideal, we obtain that $I + J = I$. Therefore we can define the maximal K_1 -nilpotent ideal as the sum of all the K_1 -nilpotent ideals in L and call it the K_1 -nilradical. Notice that, the K_1 -nilradical do not possess the properties of the radical in the sense of Kurosh.

Using the same argumentation as in the proof of Proposition 4.4 and Theorem 4.5 the following statements can be established.

Proposition 4.9. *Let I be an ideal of a Leibniz n -algebra L . Then for any $d \in \text{Der}(L)$ we have $(d(I))^{[n]} \subseteq I + d^n(I^{[n]})$ for all $n \in \mathbb{N}$.*

Theorem 4.10. *Let J be the K_1 -nilradical of a Leibniz n -algebra L . Then for any $d \in \text{Der}(L)$ we have $d(J) \subseteq J$.*

Analogously, we can establish similar results concerning the nilpotency and s -nilpotency.

By induction it is not difficult to show that the sum of s -nilpotent (nilpotent) ideals of Leibniz n -algebra L is also s -nilpotent (nilpotent, respectively) ideal of L .

Now let N be a maximal s -nilpotent (nilpotent) ideal in a finite dimensional Leibniz n -algebra L and let M be an arbitrary s -nilpotent ideal of L . Then $N + M$ is also s -nilpotent (nilpotent, respectively) and $N + M \supseteq N$. Since N is maximal s -nilpotent (nilpotent, respectively) ideal, we obtain $N + M = N$. Therefore we can define the

maximal s -nilpotent (nilpotent, respectively) ideal as the sum of all the s -nilpotent (nilpotent, respectively) ideals in L and call it the s -nilradical (nilradical, respectively).

Proposition 4.11. *Let J be the s -nilradical (nilradical) of a finite dimensional Leibniz n -algebra L over a field \mathbb{K} of characteristic zero. Then $(J + d(J))^{<m>s} \subseteq J^{<m>s} + (d(J))^{<m>s}$ $\left((J + d(J))^m \subseteq J^m + (d(J))^m, \text{ respectively} \right)$.*

Proof. Analogous to the proof of Proposition 4.4. \square

Theorem 4.12. *Let J be the s -nilradical (nilradical) of a finite dimensional Leibniz n -algebra L over a field \mathbb{K} of characteristic zero. Then $d(J) \subseteq J$ for any $d \in \text{Der}(L)$.*

Proof. Analogous to the proof of Theorem 4.5. \square

5. NON-CONJUGACY OF CARTAN SUBALGEBRAS

In this section we consider Cartan and Frattini subalgebras of Leibniz n -algebras.

Definition 5.1 ([2]). *A subalgebra C of a Leibniz n -algebra L is said to be Cartan subalgebra if*

- a) C is 1-nilpotent;
- b) $C = N_1(C)$.

The importance of considering 1-normalizer in the definition of Cartan subalgebras was shown in [1].

Proposition 5.2 ([2]). *Let C be a nilpotent subalgebra of a Leibniz n -algebra L . Then C is a Cartan subalgebra if and only if it coincides with L_0 in the Fitting decomposition of the algebra L with respect to $R(C)$.*

Similarly as in [13], if L is a direct sum of Leibniz n -algebras L_i , $1 \leq i \leq k$, and C_i are Cartan subalgebras of L_i , then $C = \bigoplus_{i=1}^k C_i$ is a Cartan subalgebra of L and any Cartan subalgebra of L has the same form.

The following result concerning the regular elements of a Leibniz n -algebra was established in [2]:

Theorem 5.3 ([2]). *Let L be a Leibniz n -algebra over an infinite field and let x be a regular element for L . Then the Fitting null-component L_0 with respect to the operator $R(x)$ is a 1-nilpotent subalgebra of L .*

In Leibniz algebras and Lie n -algebras the corresponding theorem states that L_0 is a Cartan subalgebra. However, in [2] we give an example of a Leibniz n -algebra in which this result is not true. Now we establish this result under some restrictions.

Proposition 5.4. *Let L be a Leibniz n -algebra over an infinite field and let $x = (x_2, \dots, x_n) \in L^{\times(n-1)}$ be a regular element for L such that $x_2, \dots, x_n \in L_0(R(x_2, \dots, x_n))$. Then the Fitting null-component L_0 with respect to operator $R(x)$ is a Cartan subalgebra of L .*

Proof. Due to the previous theorem, we need to prove $N_1(L_0) = L_0$. Let $y \in N_1(L_0)$. Then $[y, x_2, \dots, x_n] \in [y, L_0, \dots, L_0] \subseteq L_0$. Hence $y \in L_0$. Therefore, $N(L_0) \subseteq L_0$ and since L_0 is a subalgebra $N_1(L_0) \supseteq L_0$. Thus, $L_0 = N_1(L_0)$ and L_0 is a Cartan subalgebra of L . \square

Now let us construct a Leibniz n -algebra L such that the quotient n -algebra L/I is a simple Lie n -algebra, where

$$I = \text{ideal } \langle [x_1, \dots, x_i, \dots, x_j, \dots, x_n] \mid \exists i, j : x_i = x_j \rangle$$

is an ideal of L .

Example 5.5. Let $\{e_1, \dots, e_{n+1}, x_1, \dots, x_m\}$ be a basis of L .

Consider an algebra with the following multiplication:

$$\begin{aligned} [e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1}] &= e_i \\ [x_k, e_j, \dots, e_j] &= \alpha_{kj} x_k, \end{aligned}$$

where $1 \leq i, j \leq n+1$, $1 \leq k \leq m$, $|\alpha_{k1}|^2 + \dots + |\alpha_{kn+1}|^2 \neq 0$ for all k , and the multiplication is skew symmetric in all the variables on $\langle e_1, \dots, e_{n+1} \rangle$.

Then this algebra is a Leibniz n -algebra and $I = \langle x_1, \dots, x_m \rangle$.

Note that L/I is a simple Lie n -algebra and by [4, Theorem 2.2] we have that $F(L/I) = 0$. Hence $F(L) \subseteq I$.

Proposition 5.6. In Example 5.5, $F(L) = 0$.

Proof. Consider the subspaces

$$L_k = \langle e_1, \dots, e_{n+1}, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m \rangle, \quad 1 \leq k \leq m.$$

From the multiplication table we get that they are subalgebras. Since the dimension of these subalgebras is $n+m = \dim L - 1$, they are maximal subalgebras.

Hence, $F(L) \subseteq \bigcap_{k=1}^m L_k = \langle e_1, \dots, e_{n+1} \rangle$. But $F(L) \subseteq I = \langle x_1, \dots, x_m \rangle$. Thus $F(L) = 0$. \square

Below, we present a more general construction.

Let us consider an arbitrary Lie n -algebra with the basis e_1, \dots, e_{n+1} and the conditions

$$[e_i, f_2, \dots, f_n] \in \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1} \rangle,$$

for all $f_2, \dots, f_n \in \{e_1, \dots, e_{n+1}\}$, $1 \leq i \leq n+1$.

One of the Lie n -algebras with these conditions is a simple Lie n -algebra. Complement this algebra with independent vectors x_1, \dots, x_m with the following multiplication

$$[x_k, e_p, \dots, e_p] = \alpha_{kp}^1 x_1 + \alpha_{kp}^2 x_2 + \dots + \alpha_{kp}^m x_m$$

for all $1 \leq k \leq m$, $1 \leq p \leq n+1$. Checking identity (1) we will find restrictions on the coefficients α_{ij}^k :

$$\sum_{i=1}^m \alpha_{kp}^i \alpha_{iq}^j = \sum_{i=1}^m \alpha_{kq}^i \alpha_{ip}^j$$

for all $1 \leq k, j \leq m$, $1 \leq p, q \leq n+1$.

Hence, the satisfaction of the above condition guaranties that the supplemented algebra is a Leibniz n -algebra.

Particularly, in this way, one can supplement simple Lie n -algebras till Leibniz n -algebras.

On the ground of Example 5.5 we give the following

Example 5.7. Let L_s ($1 \leq s \leq n+1$) be a Leibniz n -algebra with the basis $\langle e_1, e_2, \dots, e_{n+1}, x_1, \dots, x_m \rangle$ and the following multiplication:

$$\begin{aligned} [e_1, \dots, e_{p-1}, e_{p+1}, \dots, e_{n+1}] &= e_p, & 1 \leq p \leq n+1, \\ [x_k, e_k, e_k, \dots, e_k] &= x_k, & 1 \leq k \leq s, \\ [x_{s+i}, e_s, e_s, \dots, e_s] &= x_{s+i}, & 1 \leq i \leq m-s, \end{aligned}$$

where the multiplication is skew symmetric in all the variables on $\langle e_1, e_2, \dots, e_{n+1} \rangle$.

Then

$$\begin{aligned} H_1 &= \langle e_1, e_2, \dots, e_s, e_{s+1}, \dots, e_{n-1} \rangle \\ H_2 &= \langle e_1, e_2, \dots, e_s, e_{s+2}, \dots, e_{n-1}, e_n \rangle \\ H_3 &= \langle e_1, e_2, \dots, e_s, e_{s+3}, \dots, e_{n-1}, e_{n+1} \rangle \end{aligned}$$

are $n-1$ dimensional Cartan subalgebras.

The subalgebras

$$\begin{aligned} N_1 &= \langle x_1, e_2, e_3, \dots, e_n \rangle \\ N_2 &= \langle e_1, x_2, e_3, \dots, e_n \rangle \\ &\vdots \\ N_{s-1} &= \langle e_1, \dots, e_{s-2}, x_{s-1}, e_s, \dots, e_n \rangle \end{aligned}$$

are n dimensional Cartan subalgebras.

The subalgebras

$$\begin{aligned} M_1 &= \langle e_1, e_2, \dots, e_{s-1}, e_{s+1}, e_{s+2}, \dots, e_n, x_s, x_{s+1}, \dots, x_m \rangle \\ M_2 &= \langle e_1, e_2, \dots, e_{s-1}, e_{s+2}, e_{s+3}, \dots, e_{n+1}, x_s, x_{s+1}, \dots, x_m \rangle \end{aligned}$$

are $m+n-s$ dimensional Cartan subalgebras.

$$\begin{aligned} C_1 &= \langle x_1, e_2, \dots, e_{s-1}, e_{s+1}, \dots, e_{n+1}, x_s, x_{s+1}, \dots, x_m \rangle \\ C_2 &= \langle e_1, x_2, \dots, e_{s-1}, e_{s+1}, \dots, e_{n+1}, x_s, x_{s+1}, \dots, x_m \rangle \\ &\vdots \\ C_{s-1} &= \langle e_1, \dots, e_{s-2}, x_{s-1}, e_{s+1}, \dots, e_{n+1}, x_s, x_{s+1}, \dots, x_m \rangle \end{aligned}$$

are $m+n-s+1$ dimensional Cartan subalgebras.

In the considered Leibniz n -algebra we found Cartan subalgebras of dimensions $n-1, n, n+m-s$ and $n+m-s+1$. Hence, in general, Cartan subalgebras of a given Leibniz n -algebra are not conjugated.

Here we give a theorem that establishes the conjugacy of Cartan subalgebras under some restrictions on the Leibniz n -algebra.

Theorem 5.8. Let L be a finite dimensional Leibniz n -algebra and H be a Cartan subalgebra of L . Suppose that

- (i) the multiplication is skew symmetric in the first two variables; and that
- (ii) for any element $h = (h_1, \dots, h_{n-1}) \in H^{\times(n-1)}$, we have $h_i \in \text{Ker } R(h)$ for all $1 \leq i \leq n-1$.

Then there is a regular element $h \in H^{\times(n-1)}$ such that $H = L_0(R(h))$.

Proof. Suppose that H is a Cartan subalgebra of a Leibniz n -algebra and $L = L_0 \oplus L_{\alpha_1} \oplus \dots \oplus L_{\alpha_s}$ is the decomposition of L into a direct sum of root subspaces with respect to H and $\Delta = \{\alpha_1, \dots, \alpha_s\}$ is the set of non-zero roots of H in L . Then the functions α_i

are multilinear and, in particular, polynomial. Since $H^{\times(n-1)}$ is an irreducible variety, it follows that $\alpha_1\alpha_2\cdots\alpha_s$ is also a non-zero polynomial function from H^{n-1} to the ground field of the Leibniz n -algebra. Hence, $\alpha_1(h^0)\alpha_2(h^0)\cdots\alpha_s(h^0) \neq 0$ for some $h^0 = (h_0, h_1^0, \dots, h_{n-2}^0)$. This means that the characteristic roots of the restriction $\overline{R}(h^0)$ of the endomorphism $R(h^0)$ to $L_1(R(h^0)) = \sum_{\alpha \in \Delta} L_\alpha$ are all nonzero, and hence $\overline{R}(h^0)$ is a non-degenerate operator.

The proof of the theorem is based on the proof of conjugacy of Cartan subalgebras in Lie n -algebras given by Kasymov [13]. Similarly, we define a polynomial function P on L by

$$P(x) = \exp R(x_1, h_1^0, \dots, h_{n-2}^0) \cdots \exp R(x_s, h_1^0, \dots, h_{n-2}^0)(h),$$

where $x = h + x_1 + \cdots + x_s, h \in H = L_0(R(h^0)), x_i \in L_{\alpha_i}$.

Notice that, if a right multiplication operator $R(x)$ is nilpotent, then $\exp R(x)$ is an inner automorphism of the Leibniz n -algebra L . Automorphisms of this kind generate a certain subgroup G_0 in the group $G = \text{Aut } L$. Elements of G_0 are called *special (invariant) automorphisms*.

Using the skew symmetrical property of the multiplication in the first two variables, we establish that the differential $d_{h^0}P$ of P at a point h^0 is an epimorphism. Hence, by facts from algebraic geometry in [13], this polynomial function P is dominating, i.e. for any non-zero polynomial function f on L there exists a non-zero polynomial function g on L such that every $y \in L$ with $g(y) \neq 0$ is represented as $y = P(x)$, where $f(x) \neq 0$.

Assuming that for any regular element $h = (h_1, \dots, h_{n-1}) \in H^{\times(n-1)}$ we have $h_i \in \text{Ker } R(h)$ for all $1 \leq i \leq n-1$, then

$$P(h_i) = \left(\prod_{j=1}^s \exp R(x_j, h_1, \dots, h_i, \dots, h_{n-2}) \right) (h_i) = h_i.$$

Hence, we can use similar induction as in [13] to prove the existence of a regular element $h \in H^{\times(n-1)}$ such that $H = L_0(R(h))$. \square

Under the conditions of Theorem 5.8 the following theorem can be proved similarly as in the case of Lie n -algebras [13].

Theorem 5.9. *Let L be a finite-dimensional Leibniz n -algebra which satisfies the conditions (i)-(ii) of Theorem 5.8. If H and K are Cartan subalgebras of L , then there exists a special automorphism δ of L such that $H = \delta(K)$.*

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